MATH 135 — QUIZ 2 — JAMES HOLLAND
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Question 1. Consider the function f defined by

$$f(x) = \begin{cases} (\sin x)/x & \text{if } x > 0 \\ 3 & \text{if } x = 0 \\ e^{-x} & \text{if } x < 0. \end{cases}$$

(i) Evaluate $f(0)$.
(ii) Evaluate $\lim_{x\to 0} f(x)$ if it exists.

Proof .:.

(i) $f(0) = 3$.
(ii) $f(x) = e^{-x}$ when $x < 0$, so $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-x} = e^0 = 1$.
Similarly, $f(x) = (\sin x)/x$ when $x > 0$, and so $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (\sin x)/x = 1$.
Hence the left and right-sided limits of f are both 1, and so the limit $\lim_{x\to 0} f(x)$ exists and is equal to 1.

Question 2. Evaluate $\displaystyle\lim_{x\to 0} \frac{(x+2)\left(\sqrt{x^2+1}-1\right)}{x^2+2x}$.

Proof .:.

We have the following equalities whenever $x^2 + 2x \neq 0$ (i.e. for $x \neq 2$ and $x \neq 0$):

$$\frac{(x+2)\left(\sqrt{x^2+1}-1\right)}{x^2+2x} = \frac{\sqrt{x^2+1}-1}{x} = \frac{\left(\sqrt{x^2+1}-1\right)\left(\sqrt{x^2+1}+1\right)}{x\left(\sqrt{x^2+1}+1\right)} = \frac{|x^2+1|-1}{x\left(\sqrt{x^2+1}+1\right)}.$$

Since $x^2 + 1$ is always positive, this absolute value doesn't affect anything: $x^2 + 1 = |x^2 + 1|$. Thus this is

$$\frac{x^2+1-1}{x\left(\sqrt{x^2+1}+1\right)} = \frac{x^2}{x\left(\sqrt{x^2+1}+1\right)} = \frac{x}{\sqrt{x^2+1}+1}.$$

So taking a limit yields

$$\lim_{x\to 0} \frac{(x+2)\left(\sqrt{x^2+1}-1\right)}{x^2+2x} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+1}+1} = \frac{0}{2} = 0.$$

Question 3. Draw the graph of a function f such that

- $f(2)$,
- $\lim_{x\to 2^-} f(x)$, and
- $\lim_{x\to 2^+} f(x)$

all exist and are all different from each other.

Proof .:.

There are infinitely many such graphs. Below is one example. Here, $f(2) = 2$ while $\lim_{x\to 2^-} f(x) = 1$ and $\lim_{x\to 2^+} f(x) = 3$.